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Inversely, to find $(D)^m$ in terms of D^m, D^{m-1}, \dots, D , we have m equations linear in $(D)^m, (D)^{m-1}, \dots, (D)$ (obtained by assuming for m in (1) the values 1, 2, 3, ... m) from which, if $(m-k)_i$ = the sum of the products of the natural numbers from 1 to $m-k$ inclusive taken i at a time,

$$(D)^m = \begin{vmatrix} D^m & -(m-1)_1 & (m-1)_2 & \dots & \mp(m-1)_{m-1} \\ D^{m-1} & 1 & (m-2)_1 & \dots & \pm(m-2)_{m-2} \\ D^{m-2} & 0 & 1 & \dots & \mp(m-3)_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D & 0 & 0 & \dots & 1 \end{vmatrix}$$

By another familiar process applied to the symbols $D^m [= (D)^{m'}]$ and $(D)^m$ we get also

$$(D)^m = \sum \frac{D^i 0^m}{i!} D^i.$$

Examples of (1).—1. To change the independent variables in D^m from

$\alpha_1, \alpha_2, \dots$ to $\theta_1, \theta_2, \dots$, where $\alpha_1 = \varepsilon^{\theta_1}, \alpha_2 = \varepsilon^{\theta_2}, \dots$; so that

$$\alpha_1 \frac{d}{d\alpha_1} = \frac{d}{d\theta_1}, \alpha_2 \frac{d}{d\alpha_2} = \frac{d}{d\theta_2}, \dots \quad \left(\left(\alpha_1 \frac{d}{d\alpha_1} + \alpha_2 \frac{d}{d\alpha_2} + \dots \right) \right)^{m'}$$

becomes by the transformation

$$\left(\frac{d}{d\theta_1} + \frac{d}{d\theta_2} + \dots \right)^{m'}. \quad \therefore \text{by (1)} \quad D^m = \left(\frac{d}{d\theta_1} + \frac{d}{d\theta_2} + \dots \right)^{m'}.$$

For a special case see Todhunter's Dif. Cal. Art. 208.

2. By Euler's theorem concerning a homogeneous function $\varphi(\alpha_1, \alpha_2, \dots)$, of n dimensions, $F((D))\varphi = F(n)\varphi; \therefore (D)^{m'}\varphi = n^{m'}\varphi; \therefore$ by (1)

$$D^m\varphi = n^{m'}\varphi.$$

SOLUTION OF A PROBLEM.

BY PROF. E. W. HYDE, UNIVERSITY OF CINCINNATI.

Problem —To show that $\cos^p \varphi \sin^q \varphi$ can be expanded into a series of *cosines* of multiples of φ when q is *even*, and into a series of *sines* of multiples of φ when q is *odd*.

First, suppose q even and $= 2n$, say. Then

$$\begin{aligned} \cos^p \varphi \sin^{2n} \varphi &= \cos^p \varphi (\sin^2 \varphi)^n = \cos^p \varphi (1 - \cos^2 \varphi)^n \\ &= \cos^p \varphi - n \cos^{p+2} \varphi + \frac{n(n-1)}{2!} \cos^{p+4} \varphi - \&c. \end{aligned}$$

Each term of this series can be expanded into a series of cosines of multiples of φ .

Second, suppose q odd and $= 2n + 1$ say, and also let p be even and $= 2m$. Then

$$\begin{aligned}\cos^{2m}\varphi \sin^{2n+1}\varphi &= (\cos^2\varphi)^m \sin^{2n+1}\varphi = \sin^{2n+1}\varphi (1 - \sin^2\varphi)^m \\ &= \sin^{2n+1}\varphi - m \sin^{2n+3}\varphi + \frac{m(m-1)}{2!} \sin^{2n+5}\varphi - \&c.\end{aligned}$$

Since each term of the right hand member is an odd power of $\sin \varphi$, each term may be expanded into a series of sines of multiples of φ .

Finally let $q = 2n + 1$, and $p = 2m + 1$ so that both are odd. Then, if we suppose $m < n$,

$$\begin{aligned}\cos^{2m+1}\varphi \sin^{2n+1}\varphi &= (\sin \varphi \cos \varphi)^{2m+1} (\sin^2\varphi)^{n-m} \\ &= \left(\frac{1}{2}\right)^{m+n+1} (\sin 2\varphi)^{2m+1} (1 - \cos 2\varphi)^{n-m};\end{aligned}$$

while if we suppose $m > n$ we have

$$\begin{aligned}\cos^{2m+1}\varphi \sin^{2n+1}\varphi &= (\sin \varphi \cos \varphi)^{2n+1} (\cos^2\varphi)^{m-n} \\ &= \left(\frac{1}{2}\right)^{m+n+1} (\sin 2\varphi)^{2n+1} (1 + \cos 2\varphi)^{m-n}.\end{aligned}$$

From the similarity of these two forms it evidently makes no difference whether m be greater or less than n , we will use then the first expression, i. e., regard m as less than n .

$$\begin{aligned}\therefore \cos^{2m+1}\varphi \sin^{2n+1}\varphi &= \left(\frac{1}{2}\right)^{m+n+1} (\sin 2\varphi)^{2m+1} \\ &\quad \times \left[1 - (n-m) \cos 2\varphi + \frac{(n-m)(n-m-1)}{2!} \cos^2 2\varphi - \&c. \right].\end{aligned}$$

Now the product of $(\sin 2\varphi)^{2m+1}$ into any even power of $\cos 2\varphi$ comes under the second case, and we have therefore only to consider the products containing *odd* powers.

The exponent of the highest odd power will be either $n-m$ or $n-m-1$; call it $2m' + 1$: then treating that product as before we have, supposing $m' < m$,

$$(\cos 2\varphi)^{2m'+1} (\sin 2\varphi)^{2m+1} = \left(\frac{1}{2}\right)^{m+m'+1} (\sin 4\varphi)^{2m'+1} [1 - \cos 4\varphi]^{m-m'}.$$

Repeating this process successively we may finally obtain either

$$\sin[2^r\varphi] [1 \mp \cos(2^s\varphi)] = \sin(2^r\varphi) \mp \frac{1}{2} \sin(2^{r+1}\varphi),$$

or

$$\sin(2^s\varphi) \cos(2^s\varphi) = \frac{1}{2} \sin(2^{s+1}\varphi).$$

To show that this will be the result we will find the degrees of the successive left hand members. That of the first is

$$2m+1+2n+1 = 2(m+n+1).$$

That of the second, supposing $n-m$ to be odd, is

$$2m+1+n-m = m+n+1.$$

That of the third, supposing $m-m'$ odd, is

$$n-m+m - \frac{n-m-1}{2} = \frac{n+m+1}{2}.$$

Thus the degree each time is half what it was before. If $n-m, m-m', m'-m'',$ etc., be *even*, we shall have for the successive degrees,

$$\begin{aligned} \text{1st, } 2m+1+2n+1 &= 2(m+n+1), \\ \text{2nd, } 2m+1+2m'+1 &= 2(m'+m+1) = m+n, \\ \text{3rd, } 2m'+1+2m''+1 &= 2(m''+m'+1) = \frac{1}{2}(m+n-2), \\ \text{4th, } 2m''+1+2m''' +1 &= 2(m''' +m''+1) = \frac{1}{4}(m+n-2), \text{ etc.} \end{aligned}$$

After the second, the degree is each time half the preceding. If the exponents $n-m, m-m',$ etc., were some even and some odd the result would be between these two.

A numerical example is added.—Let $m = 9$ and $n = 14$. Then

$$\begin{aligned} \cos^{19}\varphi \sin^{29}\varphi &= (\tfrac{1}{2})^{24}(\sin 2\varphi)^{19}[1-\cos 2\varphi]^5, \\ (\cos 2\varphi)^5(\sin 2\varphi)^{19} &= (\tfrac{1}{2})^{12}(\sin 4\varphi)^5[1-\cos 4\varphi]^7, \\ (\cos 4\varphi)^7(\sin 4\varphi)^5 &= (\tfrac{1}{2})^6(\sin 8\varphi)^5[1+\cos 8\varphi], \\ \cos 8\varphi (\sin 8\varphi)^5 &= (\tfrac{1}{2})^3(\sin 16\varphi)[1-\cos 16\varphi]^2, \\ \cos 16\varphi \sin 16\varphi &= \tfrac{1}{2} \sin 32\varphi. \end{aligned}$$

If in the expansion of the right-hand member of the first equation we take $(\cos 2\varphi)^3$ instead of $(\cos 2\varphi)^5$ we have

$$\begin{aligned} (\cos 2\varphi)^3(\sin 2\varphi)^{19} &= (\tfrac{1}{2})^{11}(\sin 4\varphi)^3[1-\cos 4\varphi]^8, \\ (\cos 4\varphi)^7(\sin 4\varphi)^3 &= (\tfrac{1}{2})^5(\sin 8\varphi)^3[1+\cos 8\varphi]^2, \\ \cos 8\varphi (\sin 8\varphi)^3 &= (\tfrac{1}{2})^2(\sin 16\varphi)[1-\cos 16\varphi], \\ &= (\tfrac{1}{2})^2[\sin 16\varphi - \tfrac{1}{2}\sin 32\varphi]. \end{aligned}$$

DISCUSSION OF AN EQUATION.

BY JOHN BORDEN, CHICAGO, ILL.

As equations of the fifth or less degree are reducible to the form

$$y^n + py + q = 0, \tag{1}$$

n being an integer and positive, its discussion is of interest.

QUESTION.—Determine the number of real roots, their limits, and sign, for all real values of p and q .

ANSWER:—I. If n is odd and p is positive, there is only one real root, whose sign is contrary to that of q and whose limits are $q \div p \times -0$, and $q \div p \times -1$.

II. If n is odd and p is negative, there are three real roots when

$$\frac{p^n}{q^{n-1}} > \frac{n^n}{(n-1)^{n-1}},$$

two of which have the same sign as q , and become equal to each other, and